



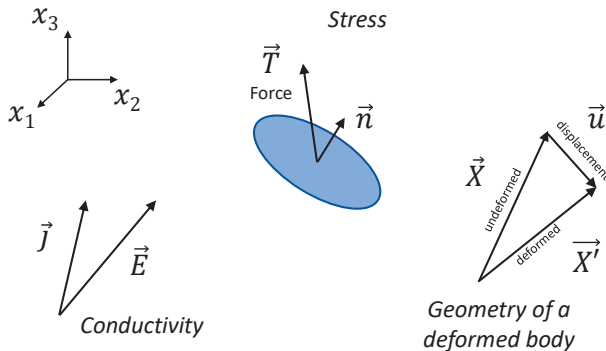
More than Math: How Tensors Help Describe Mineral Properties

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DOI: 10.2138/gselements.22.2.74

Minerals, especially crystalline ones, are generally anisotropic, meaning that some of their properties depend on direction. A spectacular example is cordierite, which, depending on the direction in which it is viewed in transmission, will appear purple or yellow.

Some quantities, such as density (ρ) or temperature (T), are not linked to a notion of direction. In this case, they are simply described by a number, a scalar. Others, however, such as the magnetic field (\vec{H}), electric field (\vec{E}), or gravity field (\vec{g}), contain not only a notion of magnitude, but also of direction. They are described by a vector determined, in a given reference system, by its components. For example, for the electric field: $\vec{E}(E_1, E_2, E_3)$ where E_1, E_2 , and E_3 are the components of the \vec{E} vector projected onto the Ox_1, Ox_2 , and Ox_3 axes of the reference frame under consideration.



If this electric field \vec{E} is applied to a mineral, it is likely to set charged particles in motion. In an isotropic solid, we expect the movement of charged particles to follow the direction of the electric field, giving rise to an electric current density E that is proportional to the magnitude of the applied electric field (this is Ohm's law, see Pommier et al. 2026 this issue). The proportionality coefficient, which is a scalar, is called conductivity. In an anisotropic crystal, however, charges may be set in motion in directions other than that of the applied electric field. This is because the response of the crystal depends not only on the direction of the applied field, but also on how this field is oriented with respect to the crystallographic directions (or planes) of the mineral: the same field can produce different currents depending on how it interacts with the crystal structure. The electric current density vector \vec{j} that describes these charge movements may therefore have a direction different from that of the electric field. To write Ohm's law in this case, which describes

a linear response (electric current density) to the driving force (electric field), we write that each component of the vector \vec{j} is a linear combination of the components of the vector \vec{E} :

$$\begin{aligned} j_1 &= \sigma_{11}E_1 + \sigma_{12}E_2 + \sigma_{13}E_3 \\ j_2 &= \sigma_{21}E_1 + \sigma_{22}E_2 + \sigma_{23}E_3 \\ j_3 &= \sigma_{31}E_1 + \sigma_{32}E_2 + \sigma_{33}E_3 \end{aligned} \quad (1)$$

The linear property of electrical conductivity is now described using nine coefficients which can be conveniently represented within a square array:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad (2)$$

This set of coefficients defines what is known as a second-rank tensor, whose components are written σ_{ij} . Each coefficient describes how a component of the current responds to a component of the electric field. For example, σ_{32} quantifies the displacement of charge along the direction Ox_3 in response to quantifies the displacement of charge along the direction along Ox_2 . The need for two indices here reflects a simple physical idea: one index identifies the direction of the applied field, while the other identifies the direction of the resulting current. In other words, the response depends on two directions at once. This is what is meant by a "rank-2" tensor. More generally, the rank of a tensor corresponds to the number of directions needed to describe a quantity. Scalars, which have no directional dependence, can be seen as rank-0 tensors, while vectors, which involve a single direction, are rank-1 tensors. In practice, this is reflected in the notation: the number of subscripts used to label the components corresponds to the number of directions involved, and therefore to the rank of the tensor.

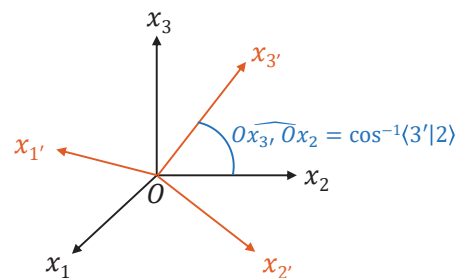
Relation (1) can be written

$$j_i = \sum_{j=1}^3 \sigma_{ij}E_j \quad (i = 1, 2, 3) \quad (3)$$

It is usual at this stage to introduce a more compact notation due to Einstein where we leave out the summation sign:

$$j_i = \sigma_{ij}E_j \quad (i, j = 1, 2, 3) \quad (4)$$

The Einstein summation convention is that when a letter suffix occurs twice in the same term, summation with respect to that suffix is to be automatically understood.



Just as the electrical conductivity tensor is used to describe the physical property that relates (linearly) the applied electric field to the induced electric current density, there are other physical properties of crystals

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that can be described by second-rank tensors. The properties described in this issue include **polarizability**^{*}, which relates the induced **dipole moment** of a material to the applied electric field as in Raman scattering (Solomatova et al. 2026 this issue), thermal conductivity, which links the applied temperature gradient to the induced heat flow density (Jackson et al. 2026 this issue), or the magnetic susceptibility, which links the applied magnetic field to the induced density of magnetization (Fu and Harrison 2026 this issue).

Although the components of the tensor depend strongly on the chosen coordinate system (and therefore vary with it), it is imperative that the property described by the tensor is invariant to the change of coordinate system.

Consider a second-rank tensor described by its A_{ij} components in the reference frame Ox_1, Ox_2, Ox_3 . In another reference frame $Ox_{1'}, Ox_{2'}, Ox_{3'}$, the same tensor will be described by another set of components $A_{i'j'}$. Knowledge of the $A_{i'j'}$ coefficients in the new coordinate system can be deduced from the A_{ij} coefficients in the old coordinate system, based on the orientation relationships between the two axis systems, and more specifically on the direction cosines

$$\langle i'|j \rangle = \cos \widehat{Ox_{i'} Ox_j} \quad (5)$$

where the bracket notation $\langle i'|j \rangle$ denotes the cosine of the angle between the directions $Ox_{i'}$ and Ox_j —that is, the scalar product between the corresponding unit vectors. This notation provides a compact way to express how directions change from one reference frame to another.

$$A_{i'j'} = \langle i'|j \rangle \langle j'|k \rangle A_{ij} \quad (6)$$

Note that this relationship includes a double dummy notation where two summations on i and j are left out.

In the above-mentioned cases, a second-rank tensor is used to describe a physical property defined by a general linear relationship between two first-rank tensors (vectors), which respectively represent a driving force and the response it induces. These tensors, which measure physical properties, must conform to the crystal symmetry. They are called *matter tensors*.

More generally, this way of linking two vectors is used in elasticity to define two important tensors: the Cauchy stress tensor and the infinitesimal **strain** tensor. The Cauchy stress tensor links the surface density of force \vec{T} at each point of a crystal to the orientation (defined by the normal vector \vec{n}) of the elementary surface to which it acts (see box in Mazzuchelli et al. 2026 this issue)

$$T_i = \sigma_{ij} n_j \quad (i, j = 1, 2, 3) \quad (7)^{**}$$

Note that an isotropic stress tensor $\sigma_{ij} = -P\delta_{ij}$ ^{***} describes the condition of hydrostatic pressure P (Mazzuchelli et al. 2026 this issue). Deformation is described from the knowledge, expressed by the displacement field vector \vec{u} , of the geometric transformation from the undeformed body (marked by a vector \vec{x}) to the deformed body (marked by a vector \vec{x}'). The components of the infinitesimal strain tensor ε_{ij} are obtained from the components of \vec{u} using the following relation

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (i, j = 1, 2, 3) \quad (8)$$

These tensors, which describe a mechanical state and not a physical property of the crystal, are not constrained by crystal symmetry. They are called *field tensors*.

* Definitions of terms presented in **blue** are given in the glossary on page 81.

** Note that it is unfortunately usual in the literature to denominate electrical conductivity and Cauchy stress by the same Greek letter, σ . We do not seek to avoid this homography here; context is key.

*** δ_{ij} is the Kronecker delta symbol defined by $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$

The linear elastic behavior described by Hooke's law expresses a linear relationship between stress and strain, both of which are described by second-rank tensors. Hooke's law, which expresses a linear relationship between two second-rank tensors, therefore involves a fourth-rank tensor

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad (9)$$

where the C_{ijkl} are the 81 stiffness constants of the crystal. Alternatively, Hooke's law can be written

$$\varepsilon_{ij} = S_{ijkl} \sigma_{kl} \quad (10)$$

where S_{ijkl} are the compliances of the crystal.

The elasticity tensor \mathbf{C} describes a property that must be invariant to changes of axes: it is a matter tensor. The law for transforming components by changing axes is given by

$$C_{i'j'k'l'} = \langle i'|i \rangle \langle j'|j \rangle \langle k'|k \rangle \langle l'|l \rangle C_{ijkl} \quad (11)$$

The compactness of Einstein's notation should not obscure the fact that relation (11) typifies 3^4 equations each with 3^4 terms on the right-hand side, making a total of 6561 terms in all.

However, the symmetries of the stress and strain tensors induce a significant redundancy in these terms. To take account of this, Voigt (1850–1919, who first introduced the *tensor* term) proposed replacing the tensor relation of Hooke's law with a simplified matrix notation. To do this we replace the two suffixes of the second-rank tensors by a single suffix running from 1 to 6 as follows

Tensor notation	11	22	33	23, 32	31, 13	12, 21
Matrix notation	1	2	3	4	5	6

This transforms second-rank tensors into vectors according to the relations

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \rightarrow \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix}; \quad \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \rightarrow \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{pmatrix} \quad (12)$$

Hooke's law can then be written in a simpler, more compact matrix form

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} \quad (13)$$

This formalism reduces the number of elastic constants to 36, and even to 21 because the (C_{ij}) matrix is symmetrical. It is important to emphasize that despite their appearance with two suffixes, the C_{ij} are not the components, and so do not transform like the components of a second-rank tensor. To transform them to other axes, it is necessary to go back to the tensor notation.

However, the matrix of elastic constants corresponds to a crystal property, and taking symmetries into account reduces the number of constants. Only triclinic crystals require the use of 21 independent constants. This number is reduced to 13 for monoclinic crystals, 9 for orthorhombic crystals, 7 (or 6) for tetragonal or trigonal crystals, 5 for hexagonal crystals, and 3 for cubic crystals. In the case of isotropic media, two elastic constants are sufficient to describe Hooke's law.

We may now link the **macroscopic perspective** (Hooke's law) to the **microscopic view** by introducing the dynamical matrix M_{ik} , a second-rank tensor describing how quantized lattice vibrations (phonons) change with direction and displacement (strain). The specific case we

consider here is the elegant connection to Hooke's law: acoustic vibrational modes in the long-wavelength limit correspond to crystal strains. Recalling Eq. (5) and that in this case, these strains arise from acoustic modes, the strain waves propagate as

$$\varepsilon_{ij}(t) = \frac{1}{2} (u_i(t)k_j + u_j(t)k_i) \quad (14),$$

where k represents the wave vector (i.e., direction in the crystal). Using Newton's equation of motion and ρ for density

$$\rho\omega^2 u_i(t) = \sum_{j,k,l} C_{ijkl} k_j k_k u_l(t) = M_{ik} u_i(t) \quad (15),$$

the symmetric dynamical matrix now has a form with familiar elastic constants

$$M_{ik} = \begin{pmatrix} C_{11}k_1^2 + C_{44}(k_2^2 + k_3^2) & (C_{12} + C_{44})k_1 k_2 & (C_{12} + C_{44})k_1 k_3 \\ (C_{12} + C_{44})k_1 k_2 & C_{11}k_2^2 + C_{44}(k_1^2 + k_3^2) & (C_{12} + C_{44})k_2 k_3 \\ (C_{12} + C_{44})k_1 k_3 & (C_{12} + C_{44})k_2 k_3 & C_{11}k_3^2 + C_{44}(k_1^2 + k_2^2) \end{pmatrix} \quad (16).$$

In addition to its link to elasticity, the dynamical matrix is of central importance for describing optical spectra (Solomatova et al. 2026 this issue) and understanding thermal transport (Jackson et al. 2026 this issue).

In this brief overview, elasticity took us directly from second-rank tensors to a fourth-rank tensor, while creating a link to the microscopic view (lattice vibrations) from the macroscopic perspective (Hooke's law). Finally, we may consider an example of a third-rank tensor, piezoelectricity, using quartz as an example. Applying a stress field (described by a second-rank tensor) to a quartz crystal causes polarization to occur, which is characterized by a vector (i.e., a first-rank tensor P_i). The linearity of this response is then described by the tensor relation

$$P_i = d_{ijk} \sigma_{jk} \quad (i = 1, 2, 3) \quad (17),$$

which introduces a third-rank piezoelectric tensor. The 27 coefficients d_{ijk} are the piezoelectric moduli. These coefficients change when the coordinate system is changed in a similar way to that seen previously

$$d_{i'j'k'} = \langle i' | i \rangle \langle j' | j \rangle \langle k' | k \rangle d_{ijk} \quad (18).$$

The situation described above corresponds to the so-called direct piezoelectric effect. Alternatively, the application of an electric field to a piezoelectric crystal induces deformation. This is known as the converse piezoelectric effect, which writes

$$\varepsilon_{jk} = d_{ijk} E_i \quad (i, j = 1, 2, 3) \quad (19)$$

with the same coefficients as for the direct piezoelectric effect.

In this *Toolkit*, we have emphasized the ability of tensors to describe the linear response of anisotropic media to several types of driving forces. In this respect, the invariance of the tensor object to coordinate changes is a fundamental property for which this mathematical formalism is particularly well suited. The scope of tensors, however, extends far beyond this presentation. They are widely used in theoretical physics, engineering, image processing, artificial intelligence, and many other scientific fields. They are used to represent and manipulate physical quantities, force fields, images, structured data, and more.

For further information on these concepts, readers are invited to refer to J.F. Nye's book "Physical Properties of Crystals," which has been the undisputed reference on the subject since its first edition in 1957, and to M.T. Dove's book "Introduction to Lattice Dynamics."

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